THE NON-EUCLIDEAN GEOMETRY INEVITABLE.

IN saying that, at a certain stage of civilisation or general mental enlightenment, the doctrine of the conservation of matter appears of necessity, nothing need be maintained about the finality of that doctrine.

So also in regard to the conservation of energy. When general science and precision of measurement had reached a certain stage of development, a certain perfection, this question of the seeming disappearance of accurately estimated energy could no longer be overlooked. The amount of conscious or unconscious dodging required to avoid this consideration becomes too laborious, and Mayer, Colding, Grove, Helmholtz, Joule, and a host of others find the doctrine of the conservation of energy forced upon them.

Again, without considering as irreducible this foundation-stone of science, we can see that it makes untenable the realistic materialism for which the brain secretes thought as the liver secretes bile; and equally untenable the Scotch realistic dualism of an immaterial soul using the brain.

For suppose all the natural forces of the universe so constituted and connected that one may pass into another in accordance with certain definite ratios of equivalence, such as Joule's mechanical equivalent of heat, but that, so reckoned, they can neither be increased nor diminished as a whole, any more than can the finite amount of matter.

If, now, mind is a piece of the material world; if what we call mental energy, while mental, is yet a part of the sum of physical energy, then some of this invariable quantity of energy exists from time to time as mental energy, and so we would expect to be able to

say that a certain amount of chemical energy disappears, but reappears as mental energy, or perhaps disappears as mental energy but reappears as mechanical energy or heat. But the absolute tests of science would demonstrate that such is never the case. No bit of physical energy ever disappeared as physical energy to become even for an instant mental energy. There is not a single point in the series of changes which take place in the brain at which all the energy is not in actual existence as physical energy. There is not a point where anything of the nature of thought could be inserted as a possible link in the chain of transformations of energy. Materialism and dualism are equally impossible. Idealistic monism becomes inevitable. Thinking mathematicians have long known that number is wholly of human make, and agree that the idea of time has no essential connexion with it. The question of the subjectivity of space is as natural as the question of the actual existence of boundaries. I was an interested listener to a debate, between a chemist and a metaphysician, as to the existence of a boundary be tween the black and the white half of a surface which was before them. The chemist said he thought of the white part, and then of the black part, but never of anything between them. To him the idea of a boundary absolutely without any breadth, and belonging as much to the white as to the black, appeared highly artificial, and utterly uncalled for. To the metaphysician the common boundary, the line between the white and black, appeared more real than the colors it bounded. The line without width was just what his mind took hold of, and dwelt upon.

Is geometry then as wholly subjective as is arithmetic? It has been a product of pure logic applied to certain fundamental properties attributed primarily to the straight line, secondarily to the plane, circle, and sphere. But whence these properties, these lines, these surfaces? If we can agree upon these will all be settled?

Can any one give a descriptive definition of a straight line or a plane? Euclid's fourth definition is "A line which lies evenly between the points in itself is a straight line." His seventh is "A surface which lies evenly between the straight lines in itself is a plane."

I would paraphrase this: A straight is a line which looks the same from every point not in it. A plane is a surface which looks the same from every point not in it. But for the work of demonstration, Euclid substitutes for his pseudo-description a theorem "Two straight lines cannot enclose a space." We paraphrase this by saying, A straight is a line determined by any two points in it. But just here an interesting question has suggested itself to my mind: "Does not this modern paraphrase fail to touch one tremendously important matter covered by Euclid's Sixth Postulate? [P. Tannery gives as Postulate 6: *"Et que deux droites ne comprennent pas d'espace."* (Axiom 12 in Gregory; Axiom 9 in Heiberg).]

Space may be homogeneous and boundless (though not infinitely great), and straights may be homogeneous and boundless, and look the same from every point not in them, and each be determined by any two points in it; and yet each may be finite and all may be equal in size. But Euclid's Sixth Postulate assumes in addition that straights are not finite, since if finite and boundless two must recur to any crossing-point, and so would "enclose a space" in Euclid's sense. Did Euclid build so much better than he knew? Or was he conscious of that truth which in modern times waited for Riemann, that space may be boundless, yet finite in size?

When the French Revolution had beheaded all adherence to authority, when even the years and the months were renamed, in the *séance de l'école normale du 26 pluviose an III*, the celebrated Fourier proposed new definitions of the sphere, plane, circle, straight, as foundation for a new treatment of the beautiful science of space.

Take any two points on any solid. Let one remain at rest while the solid moves. The other describes a sphere. Two spheres intersect in a circle. If the spheres are equal and grow, this circle describes a plane. If the spheres touch and one decreases as the other grows, their point of contact describes a straight.

Monge, that delicate spirit, founder of the idea of elegance in demonstration, was present, and suggested certain objections to the views of Fourier, but neither seemed to suspect that these defini-

tions, however perfect, conduct not to Euclidean geometry, but to pangeometry.

How had Euclid managed not only to bury this immortal double-ghost of his space, but to conceal the grave for two thousand years? Euclid did not try to hide the non-Euclidean geometry. That was done by the superstitious night of the fanatic dark ages, from which night we have finally emerged, to find again what Euclid knew.

I believe the Euclid of twenty centuries before the birth of Gauss could still have taught the Gauss of 1799. Let us see. At the end of that year Gauss from Braunschweig writes to Bolyai in Klausenburg as follows:

> "I very much regret that I did not make use of our former proximity to find out *more* of your investigations in regard to the first grounds of geometry; I should certainly thereby have spared myself much vain labor, and would have become more restful than any one such as I can be, so long as, on such a subject, there yet remains so much to be wished for. In my own work thereon I myself have advanced far (though my other wholly heterogeneous employments leave me little time therefor), but *the* way, which I have hit upon, leads not so much to the goal which one wishes, as much more to making doubtful the truth of geometry. I have hit upon much which, with most, would pass for a proof, but which in my eyes proves as good as nothing. For example, if one could prove that a rectilineal triangle is possible whose content may be greater than any given surface, then am I in condition to prove with perfect rigor all geometry. Most would indeed let that pass as an axiom; I not; it might well be possible, that, how far apart soever one took the three vertices of the triangle in space, yet the content was always under a given limit. I have more such theorems, but in none do I find anything satisfying."

From this letter we see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven. For, strangely enough, though nothing renders it impossible that the space of our physical experience may be this very year satisfactorily shown to belong to Lobatschewsky or to Riemann, yet the same is not true for Euclid. To decide our space is Lobatschewsky's, one need only show a single rectilineal triangle whose angle-sum measures less than a straight angle. A single rectilineal triangle with angle-sum greater than a straight angle would give all

our space to Riemann. And either of these could be shown to exist by imperfect measurements, such as human measurements must always be. For example, if our instruments for angular measurement could be brought to measure an angle to within one millionth of a second, then if the lack or excess in the angle-sum were as great as two millionths of a second, we could make certain its existence.

But to prove Euclid's system, we must show that this angle-sum is *exactly* a straight angle, which nothing human can ever do. Euclid himself tried his own calm, immortal genius, and the genius of his race for perfection, against this angle-sum. The benign intellectual pride of the founder of the mathematical school of the greatest of universities, Alexandria, would not let the question cloak itself in the obscurities of the infinitely great or the infinitely small. He said to himself: "Can I prove this plain, straightforward, simple, if somewhat inelegant theorem: If a straight line meeting two straight lines, make those angles, which are inward and upon the same side of it, less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles." [Williamson's translation; Postulate 5, P. Tannery and Heiberg, Axiom 11 in Gregory.]

And let not the twentieth-century-American, in the insolence of his newness, underestimate the subtle power of that old Greek mind. The Chicago Fair produced no Venus of Milo. Euclid's own treatment of proportion is found as flawless in the chapter which Stolz devotes to it in 1885 as when through Newton it first gave us our present continuous number-system.

But what fortune had this genius in the fight with its self-chosen simple theorem? Was it found to be deducible from all the definitions, and the nine "Common Notions" and the five other Postulates of the immortal Elements? Not so. But meantime Euclid went ahead without it through twenty-eight propositions, more than half his first book. But at last came the practical pinch, then as now the triangle's angle-sum. He gets it by his twenty-ninth theorem: "A straight falling upon two parallel straights makes the alternate angles equal." But for the proof of this he wants that recalcitrant proposition which has so long been keeping him awake

nights and waking him up mornings. One last struggle, and then, true man of science, he acknowledges it indemonstrable, and spreads it in all its ugly length among his postulates.

But just here the modern translators miss a most charming point. With inartistic dullness, (see, e. g. Todhunter's), they cite in proposition twenty-nine the whole repulsive Postulate 5. But Euclid's delicate genius revolted at the very moment of transcription, and what he actually wrote down was something entirely different and much more elegant, rendered thus by Williamson, 1781: "those [straights] which are produced indefinitely from less than two right angles meet."

The Greek who dared answer to great King Ptolemy, "There is no royal road to geometry," dared carry out for himself the beautiful system of geometry which comes from the contradiction of his indemonstrable postulate; which exists if there be straights produced indefinitely from less than two right angles yet nowhere meeting. Moreover, since Schiaparelli has restored the astronomical system of Eudoxus, and Hultsch has published the writings of Autolycus, we see that Euclid knew surface-spherics, was familiar with triangles whose angle-sum is more than a straight angle.

Of how inevitably the three systems of geometry flow from just exactly the attempt Euclid made, the attempt to demonstrate his postulate fifth, we have a most romantic example in the work of an Italian priest, Saccheri, who died the fifth of October, 1733. He was a Jesuit of San Remo, who commenced to teach at Pavia in 1697, and died at Milan, where he directed the Collegio di Brera. He studied Euclid in the edition of Clavius, where the fifth postulate is given as Axiom 13. Saccheri says it should not be called an axiom, but ought to be demonstrated. He tries this seemingly simple task; but his work on it swells to a quarto book of 101 pages, four pages of index, and forty-eight figures; and with all that, he finds not a demonstration of the postulate, but, instead, three different systems of geometry.

His first proposition is: "I. In a quadrilateral $A\ B\ C\ D$, right-angled at A and B, and with opposite sides $A\ C$, $B\ D$ equal, the angles at C and D are equal." Then, after three more propositions

and three corollaries, he says: "Definitions. There are three hypotheses to distinguish, according to the nature of the angles C and D in proposition I: *hypothesis anguli recti, hypothesis anguli obtusi, hypothesis anguli acuti.*"

His propositions V, VI, VII are, that if either hypothesis is true in a single case, it is so always.

VIII–IX. In a right-angled triangle the sum of the oblique angles is equal to, greater than, or less than, a right angle, according as the hypothesis is *anguli recti, anguli obtusi, anguli acuti.*

XI–XII. In the first two hypotheses a perpendicular and an oblique to the same straight will meet.

XIII. In these two hypotheses Euclid's Postulate 5 is true.

XV–XVI. According as a triangle's angle-sum is equal to, greater than, or less than, a straight angle, we have *hypothesis anguli recti, obtusi, acuti.*

XVII. With hypothesis *anguli acuti* we can draw a perpendicular and an oblique to the same straight which nowhere meet. (Two solutions given.)

Let this suffice as a specimen of Saccheri's marvellous quarto. Clifford loved the hypothesis *anguli obtusi.* That great astronomer and geometer Sir Robert Ball actually believes in it. But Saccheri, like our profound American mathematician, Professor Oliver of Cornell, was powerfully drawn to the *hypothesis anguli acuti.* He fully realised the momentous consequences involved; nothing less than a new conception of nature, less mechanical than Newton's, and for a Catholic priest beyond question *unorthodox.* "He confessed to a distracting heretical tendency on his part in favor of the *hypothesis anguli acuti,* a tendency against which, however, he kept up a perpetual struggle (*diuturnum proelium*)."

The Inquisitor-general and the Archbishop of Milan saw Saccheri's book on July 13, 1733; the Provincial of the Company of Jesus on August 16, 1733. Within less than two months Saccheri was dead and buried. Not so his book. It was reviewed in the "Acta Eruditorum" in 1736. It was probably in the library at Göttingen about 1790–1800, for it is marked with an asterisk in the "Bibliotheca Mathematica" of Murhard. In this work it is signal-

ised (T. II, p. 43) among the writings consecrated to the explication, to the criticism, or to the defence of Euclid ("Einleitungs- und Erläuterungsschriften, auch Angriffe und Vertheidigungen des Euklides"). It therefore attained a certain notoriety. Did it escape the notice of Gauss? Jacobi, writing to Legendre, accuses Gauss of spreading a veil of mystery over his work.

Now, in the generation just preceding Gauss there worked a person so extraordinary that even Kant calls him *"der unvergleichliche Mann,"*—John Henry Lambert. He was the originator of Symbolic Logic. He fully recognised that the four algebraic operations, addition, subtraction, multiplication, division, have each an analogue in logic, namely, *Zusammensetzung*, *Absonderung*, *Bestimmung*, *Abstraction*, which may be symbolised by $+$, $-$, $\times$, $\div$. He also perceived the *inverse* nature of the second and fourth as compared with the first and third. He enunciates with perfect clearness the principal logical laws, such as the commutative and the distributive. He *develops* simple logical expressions precisely as Boole did later. He interpreted and represented hypothetical propositions precisely as Boole did. In one passage at least he recognised that the inverse process, marked by division, is an *indeterminate* one. Venn says: "To my thinking he and Boole stand quite supreme in this subject in the way of originality."

The problem of the arithmetical quadrature of the circle is as old as mathematics. Lambert it was who first proved the task of the π-computers endless by demonstrating that π is irrational. This alone would have made him immortal. He developed De Moivre's theorems on the trigonometry of complex variables, and introduced the hyperbolic sine and cosine, denoted by the symbols $sinh\,x$, $cosh\,x$. Now, the development of the theory of complex variables is one of the chief claims of Gauss.

In the very short and imperfect sketch of Lambert by F. W. Cornish of Eton College inserted in the "Encyclopædia Britannica" in 1882 we read:

"In Bernouilli and Hindenburg's *Magazin* (1787–1788) he treats of the roots of equations and *of parallel lines*."

From the deepest analytical mind of his generation that could only mean the non-Euclidean Geometry. The essay, "Zur Theorie der Parallellinien," was written in September, 1766, but first published in 1786 by F. Bernouilli (a kinsman of John Bernouilli) from the papers left by Lambert, and appears in the *Leipziger Magazin für reine und angewandte Mathematik*, herausgegeben von J. Bernouilli und C. F. Hindenburg, erster Jahrgang, 1786, Seite 137 ff. It is so important that the Leipziger Gesellschaft der Wissenschaften are about to issue a reprint of it in their Abhandlungen.

In this remarkable work Lambert maintains:

1) The Parallel-Axiom needs a proof since it does not hold for the geometry on a sphere.

2) In order to bring before the perceptive intuition a geometry in which the triangle's angle-sum is less than two right angles, we need the help of an "imaginary sphere."

3) In a space in which the triangle's angle-sum is different from two right angles, there is an absolute measure [a natural unit for length].

The rare copies which exist of W. Bolyai's "Tentamen Juventutem," such as that sold by Friedländer in 1884 at 120 marks, are dated 1832–1833. But W. Bolyai, in his "Kurzer Grundriss" (1851) speaks of it as "einem lateinischen Werke von 1829." In this "Latin work" he gives, attributing it to his son John, the expression for a circle in terms of its radius by means of π, e, and this natural unit.

In July, 1831, Gauss, in a letter to Schumacher, gives precisely this formula.

In 1829 Lobatschewsky published the elements of his non-Euclidean geometry in the Kasan *Messenger*.

In 1831 we see Schumacher using the hypothesis "if the geometry of Euclid be not true," and Gauss tells him later that "a certain Schweikardt has given to this geometry the name of *astral geometry*."

In 1846 Gauss writes that he had reread Lobatschewsky's "Geometrische Untersuchungen," and that "the exposition is totally different from that which I had projected."

In this very year Philip Kelland began to teach the non-Euclidean geometry to classes in the University of Edinburgh. In his paper on the subject, read December 21, 1863, before the Royal Society of Edinburgh, he says: "For the last seventeen years I have made it the repeated subject of lectures and essays in my class." Further on he says:

"Some years ago there appeared in *Crelle's Journal* a notice of a work, entitled 'Imaginary or Impossible Geometry,' viz., a discussion of the conclusions which would follow from the assumption as an axiom of the hypothesis that 'the three angles of a triangle are together less than two right angles.' I have never met with any statement of the propositions which the author deduced from this hypothesis."

I take these "some years ago" to be less than "seventeen years," and Kelland to be an independent discoverer of the inevitable non-Euclidean geometry.

Even the newness of America did not prevent our having an independent discoverer of the inevitable, namely, Prof. G. P. Young, the title of whose paper is: "The Relation Which Can Be Proved to Subsist Between the Area of a Plane Triangle and the Sum of the Angles, on the Hypothesis that Euclid's Twelfth Axiom Is False." Read before the Canadian Institute, February 25, 1860. Published in the *Canadian Journal of Industry, Science, and Art*, New Series, Vol. V, 1860, pp. 341–356. He says:

"I propose to prove in the present paper that if Euclid's Twelfth Axiom be supposed to fail in any case, a relation subsists between the area of a plane triangle and the sum of the angles. Call the area A and the sum of the angles s; a right angle being taken as the unit of measure. Then $A = k(2 - s)$; k being a constant finite quantity, that is, a finite quantity that remains the same for all triangles. This formula may be considered as holding good, even when Euclid's Twelfth Axiom is assumed to be true; only k is, in that case, infinite."

J. C. Glashan of Ottawa, Canada, assures us that "this paper was drawn up without the slightest knowledge whatsoever that anything had ever before been written or spoken on the subject."

The proof, which is in the style of Euclid, is thoroughly elementary, even more so perhaps than Bolyai's, and, like his, is applied to but two of the three geometries of space of constant curva-

ture; the assumption of Euclid's Sixth Postulate in the very first proposition, shutting out elliptic geometry. Omitting this proposi tion, the proof is easily extended to pangeometry.

In 1877 Grassmann pointed out that his "Ausdehnungslehre" of 1844 contained a complete foundation for an analytical development of non-Euclidean spaces. He gives as an example the "spherical space of Helmholtz." He mentions also Riemann's "Probevorlesung" of 1854, of which Dedekind thus describes the effect on Gauss:

> "Nun setzte ihn die Vorlesung, welche alle seine Erwartungen übertraf, in das grösste Erstaunen, und auf dem Rückwege aus der Facultäts-Sitzung sprach er sich gegen Wilhelm Weber mit höchster Anerkennung und mit einer bei ihm seltenen Erregung über die Tiefe der von Riemann vorgetragenen Gedanken aus."

Here let us pause. Our thesis is more than established. But of the many who have shown the non-Euclidean geometry *mechanically* inevitable, let me mention just one, that great astronomer and geometer, Sir Robert Stawell Ball, who wrote of late:

> "I quite agree with you as to the importance and the interest of the subject. The developments which it suggests are truly astonishing. It is also noteworthy how many mathematicians, approaching the subject from very varied sides, have been led to the study of what mathematics would be like without the eleventh axiom."

And now what is the final outcome, judged from the highest standpoint, that of a pure, fearless philosophy? It is nothing less than a new freedom to explain and understand our universe and ourselves.

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